Algebraic quantum hypergroups and duality

A. Van Daele

Department of Mathematics
KU Leuven (Belgium)

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This is part of more general work in progress with M. Landstad (NTNU Trondheim).

Introduction

Let A and B be finite-dimensional unital algebras and $(a, b) \mapsto \langle a, b \rangle$ a non-degenerate bilinear form. The product in one algebra induces a coproduct Δ on the other by

$$\langle aa', b \rangle = \langle a \otimes a', \Delta(b) \rangle \quad \text{and} \quad \langle a, bb' \rangle = \langle \Delta(a), b \otimes b' \rangle.$$

There are plenty of examples:

- A finite quantum group with its dual.
- A finite quantum groupoid with its dual.
- A finite quantum hypergroup with its dual.

Roughly speaking, for finite quantum groups (Hopf algebras), the coproducts are unital and homomorphisms, for finite quantum groupoids (weak Hopf algebras) the coproducts are homomorphisms but no longer unital and for finite quantum hypergroups, they are unital but no longer homomorphisms.

In all these cases, we have integrals and antipodes, with a unique behavior with respect to each other (see the next slide).

Proposition

Let (A, Δ) be a Hopf algebra with a left integral φ. Then

$$(\iota \otimes \varphi)((1 \otimes a)\Delta(c)) = S((\iota \otimes \varphi)(\Delta(a)(1 \otimes c)))$$

Proof.

$$\begin{aligned}(1 \otimes a)\Delta(c) &= \sum_{(a)}(S(a_{(1)})a_{(2)} \otimes a_{(3)})\Delta(c) \\ &= \sum_{(a)}(S(a_{(1)}) \otimes 1)\Delta(a_{(2)}c).\end{aligned}$$

If we apply φ on the second leg and use left invariance we obtain

$$(\iota \otimes \varphi)((1 \otimes a)\Delta(c)) = \sum_{(a)}(S(a_{(1)})\varphi(a_{(2)}c).$$

□

It is a remarkable feature that this relation of the antipode with the left integral is true also for quantum groupoids and quantum hypergroups. This observation is the motivation for a generalization we want to treat in this talk.

Involutive pairs of *-algebras

Let A and B be non-degenerate *-algebras. Consider a non-degenerate bilinear form $(a,b) \mapsto \langle a,b \rangle$ on $A \times B$. The pairing induces locally convex topologies.

Definition

The pairing is admissible if the products and the involutions are (separately) continuous.

Proposition

There exist left and right actions of A on B and of B on A defined by

$$\langle aa', b \rangle = \langle a, a' \triangleright b \rangle \qquad \langle aa', b \rangle = \langle a', b \triangleleft a \rangle$$
$$\langle a, bb' \rangle = \langle a \triangleleft b, b' \rangle \qquad \langle a, bb' \rangle = \langle b' \triangleright a, b \rangle$$

Proposition

There exists linear maps S on A and on B defined by

$$\langle a^*, b \rangle = \langle a, S(b)^* \rangle^- \quad \text{and} \quad \langle a, b^* \rangle = \langle S(a)^*, b \rangle^- .$$

We have some obvious properties of the maps S (the antipodes).

Proposition

- *For all* a, b *we have* $\langle S(a), b\rangle = \langle a, S(b)\rangle$.
- *For all* a *we have* $S(S(a)^*)^* = a$, *similarly for* S *on* B.

We can formulate relations among the actions, the involutions and the maps S.

Proposition

For all a, b *we have* $S(a)^* \triangleleft b = S(b^* \triangleright a)^*$.

Proof.

$$\langle S(a)^*, bb'\rangle^- = \langle a, (bb')^*\rangle = \langle a, b'^* b^*\rangle$$
$$= \langle b^* \triangleright a, b'^*\rangle = \langle S(b^* \triangleright a)^*, b'\rangle^-$$

□

We now add one more condition. It is the first condition that is not of a topological nature. It is not automatic, even in finite dimensions.

Definition

An admissible pair of *-algebras is called an involutive pair if the maps S are anti-isomorphisms. We call the map S a pre-antipode.

This implies that the maps $a \mapsto S(a)^*$ and $b \mapsto S(b)^*$ are conjugate linear isomorphisms.

These properties will induce some other relations among the actions and the pre-antipodes. Here is an example:

Proposition

For all a, b *we have* $S(a) \triangleleft b = S(S(b) \triangleright a)$.

Examples of involutive pairs

Example

Let G be a locally compact group. Denote by A the algebra $K(G)$ of complex functions with compact support and pointwise operations. Denote by B the space $K(G)$ with convolution product. One can easily check that the pairing given by $\langle f, g \rangle = \int f(p)g(p)\, dp$ satisfies all the requirements of an involutive pair.

Example

Let (A, Δ) be a multiplier Hopf *-algebra with positive integrals. Let B be the dual $\widehat{A}$. The duality gives a pairing of A with B making it again into an involutive pair.

A special case of this is obtained with a finite-dimensional Hopf *-algebra and its dual.

Remarks

The above framework is in a way still too restrictive. But it works fine in finite dimensions and for many other cases (see the next slide).

Operator algebraic pairs

Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two Hilbert spaces and W a bounded operator on the Hilbert space tensor product $\mathcal{H}_1 \otimes \mathcal{H}_2$. Define the subspaces A of $B(\mathcal{H}_1)$ and B of $B(\mathcal{H}_2)$ given by

$$A = \{(\iota \otimes \omega)W \mid \omega \in B(\mathcal{H}_2)_*\} \quad \text{and} \quad B = \{(\omega \otimes \iota)W \mid \omega \in B(\mathcal{H}_1)_*\}.$$

For any Hilbert space $\mathcal{H}$, we use $B(\mathcal{H})$ for the von Neumann algebra of all bounded linear operators on $\mathcal{H}$ and $B(\mathcal{H})_*$ for the space of normal linear functionals on $B(\mathcal{H})$. The maps $\iota \otimes \omega$ and $\omega \otimes \iota$ are slice maps from $B(\mathcal{H}_1 \otimes \mathcal{H}_2)$ to $B(\mathcal{H}_1)$ and $B(\mathcal{H}_2)$ resp.

We can define a non-degenerate pairing on $A \times B$ by

$$\langle a, b\rangle = \omega(b) = \omega'(a).$$

where $a = (\iota \otimes \omega)W$ and $b = (\omega' \otimes \iota)W$. Assume A and B are non-degenerate algebras.

This pairing admits actions and so it is an admissible pairing:

$$a \triangleleft b = (\iota \otimes \omega(b\,\cdot\,))W \quad \text{and} \quad b' \triangleright a = (\iota \otimes \omega(\,\cdot\, b'))W.$$

Similarly for the left and right actions of A on B .

Integrals relative to the antipode

Recall the property of a left integral on an algebraic quantum group:

$$(\iota \otimes \varphi)((1 \otimes a)\Delta(c)) = S((\iota \otimes \varphi)(\Delta(a)(1 \otimes c))).$$

If we pair this formula with an element b of the dual algebra we get

$$\varphi(a(c \triangleleft b)) = \varphi((a \triangleleft S(b))c). \tag{1}$$

Similarly when ψ is a right integral we find

$$\psi((b \triangleright a)c) = \psi(a(S(b) \triangleright c)). \tag{2}$$

We use these formulas for the definition of integrals in the case of an involutive pair (A, B).

Definition

A linear functional φ on A is a left integral if (**??**) holds for all $a, c \in A$ and $b \in B$.
A linear functional ψ on A is a right integral if (**??**) holds for all $a, c \in A$ and $b \in B$.

Properties of integrals

Proposition

It there is a faithful left integral on A, the antipode S on B satisfies $S(bb') = S(b')S(b)$. Similarly when there is a faithful right integral on B.

Proof.

Take $a, c \in A$ and $b, b' \in B$. Then

$$\begin{aligned}\varphi((a \triangleleft S(bb'))c) &= \varphi(a(c \triangleleft bb')) = \varphi(a(c \triangleleft b \triangleleft b')) \\ &= \varphi((a \triangleleft S(b'))(c \triangleleft b)) = \varphi((a \triangleleft S(b')S(b))c)\end{aligned}$$

We then use that φ is faithful and that the action is faithful. □

Since we aim at a theory with faithful integrals and duality, it is therefore natural to require that the antipodes are anti-automorphisms.

Proposition

Assume that φ is a left integral. Then ψ, defined as $\varphi \circ S$ is a right integral.

Proof.

Take $a, c \in A$ and $b \in B$. Then

$$\begin{aligned}\psi((b \triangleright a)c) &= \varphi(S((b \triangleright a)c)) = \varphi(S(c)(S(b \triangleright a))) \\ &= \varphi(S(c)(S(a) \triangleleft S^{-1}(b))) = \varphi((S(c) \triangleleft b)S(a)) \\ &= \varphi(S(S(b) \triangleright c)S(a)) = \psi(a(S(b) \triangleright c))\end{aligned}$$

□

We have used that S and anti-homomorphism of A (and of B).

What about uniqueness of integrals? What about invariance of integrals?

Uniqueness of integrals

For the proof of the uniqueness of integrals, we found inspiration in the following argument, valid in the finite-dimensional case. First observe the following.

Proposition

Assume that a non-zero linear functional φ is left invariant in the sense that $(\iota \otimes \varphi)\Delta(a) = \varphi(a)1$ for all a. Then we have $\Delta(1) = 1 \otimes 1$. Conversely, if $\Delta(1) = 1 \otimes 1$ any left integral is left invariant.

The first statement follows from coassociativity of Δ. For the second one we take $a = 1$ in

$$\varphi(a(c \triangleleft b)) = \varphi((a \triangleleft S(b))c).$$

This gives $\varphi(c \triangleleft b) = \varphi(c)\varepsilon(b)$.

Proposition

Assume that we have pair of finite-dimensional algebras A and B. Assume that φ' is left invariant and that φ is a faithful left integral. Then φ' is a scalar multiple of φ.

Proof.

We start with

$$S((\iota \otimes \varphi)(\Delta(a)(1 \otimes c))) = (\iota \otimes \varphi)((1 \otimes a)\Delta(c))$$

and apply φ'. Because $\varphi' \circ S$ is right invariant we get for the left hand side $\varphi'(S(a))\varphi(c)$. Now choose a so that $\varphi(a\,\cdot\,) = \varepsilon$. Then we get for the right hand side $\varphi'(c)$ and we see that $\varphi'(c) = \varphi'(S(a))\varphi(c)$. □

Remarks

Observe that this gives a simple proof of the uniqueness of integrals for finite-dimensional Hopf algebras.

It is not so easy and far from obvious to generalize this argument, but some of this is possible.

The dual integral

Proposition

When φ is a left integral on an algebraic quantum group (A, Δ), a right integral $\widehat{\psi}$ on the dual $(\widehat{A}, \widehat{\Delta})$ is defined by $\psi(b) = \varepsilon(c)$ when $b = \varphi(\,\cdot\, c)$ with $a \in A$.

In order to generalize this result we need the counit on A. It can be defined by $\varepsilon(b \triangleright a) = \langle a, b \rangle$ provided A is an algebra with local units.

Definition

Assume that φ is a faithful left integral on A and that all elements of B are of the form $\varphi(\,\cdot\, c)$ for some $c \in A$. Define $\widehat{\psi}(b) = \varepsilon(c)$ if $b = \varphi(\,\cdot\, c)$.

For all a we have

$$\langle a, S(b')b \rangle = \varphi((a \triangleleft S(b')c) = \varphi(a(c \triangleleft b')$$

so that $\widehat{\psi}(S(b')b) = \varepsilon(c \triangleleft b') = \langle c, b' \rangle$.

Proposition

$\widehat{\psi}$ is a right integral on B.

Proof.

If $b = \varphi(\,\cdot\, c)$ then $S(a) \triangleright b = \varphi(\,\cdot\, S(a)c)$ so that

$$\widehat{\psi}(S(b')(S(a) \triangleleft b)) = \langle S(a)c, b' \rangle.$$

On the other hand

$$\widehat{\psi}((a \triangleright S(b'))b) = \widehat{\psi}(S(b' \triangleright S(a))b) = \langle c, b' \triangleright S(a) \rangle$$

and we see that

$$\widehat{\psi}(S(b')(S(a) \triangleleft b)) = \widehat{\psi}((a \triangleright S(b'))b)$$

□

Algebraic quantum hypergroups

Definition

Let (A, Δ) be a non-degenerate *-algebra with a regular coproduct Δ that admits a counit. Assume that Δ is unital and that the counit is a homomorphism. We call (A, Δ) an algebraic quantum hypergroup if there is a faithful left integral with an antipode S that is an anti-isomorphism.

The counit, the antipode and the left integral are unique.

The dual B is defined as the space of elements $\varphi(\,\cdot\, a)$ where $a \in A$ with the obvious pairing of A and B.

Proposition

The dual is again an algebraic quantum hypergroup for the product and coproduct adjoint to the coproduct and product. The involution is defined by $\langle a, b^* \rangle = \langle S(a)^*, b \rangle^-$.

This is an example of an involutive pair as we have treated here.

An example from the bicsrossproduct theory

Let G be a group with two subgroups H and K satisfying $H \cap K = \{e\}$. Denote by Ω the set of pairs (h, k) in $H \times K$ satisfying $hk \in KH$.

Notation

For $(h, k) \in \Omega$ we write $hk = (h \triangleright k)(h \triangleleft k)$ where $h \triangleright k \in K$ and $h \triangleleft k \in H$.

Ω is a groupoid, in two ways. We use Ω and $\widehat{\Omega}$.

Proposition

- *In Ω the product $(h_1, k_1)(h_2, k_2)$ is defined as $(h_1, k_1 k_2)$ if $h_2 = h_1 \triangleleft k_1$. The inverse of (h, k) is $(h \triangleleft k, k^{-1})$.*
- *In $\widehat{\Omega}$, the product $(h_1, k_1)(h_2, k_2)$ is defined as $(h_1 h_2, k_2)$ if $k_1 = h_2 \triangleright k_2$. The inverse of (h, k) is $(h^{-1}, h \triangleright k)$.*

Theorem

The groupoid algebras $F(\Omega)$ *and* $F(\widehat{\Omega})$ *are a pair of* **-algebras for the obvious pairing*

$$\langle f, g\rangle = \sum_{(h,k)} f(h,k)g(h,k).$$

A dual pair of integrals is given by

$$\varphi(f) = \textstyle\sum_h f(h,e) \quad \text{and} \quad \widehat{\psi}(g) = \textstyle\sum_k g(e,k)$$

while for the counits we have $\varepsilon(f) = \sum_k f(e,k)$ *and* $\varepsilon(g) = \sum_h g(h,e)$*.*
We take here $f \in F(\Omega)$ *and* $g \in F(\widehat{\Omega})$*.*

The antipodes are given by the inverse $hk \mapsto (hk)^{-1}$*. So*

$$(S(f))(h,k) = f((h \triangleleft k)^{-1}, (h \triangleright k)^{-1}).$$

for $f \in F(\Omega)$*. Similarly for* $g \in F(\widehat{\Omega})$*.*

Conclusions and further research

- We have a satisfactory notion of an algebraic quantum hypergroup with duality .
- There is attempt to define topological quantum hypergroups with duality.
- In fact, quantum hypergroups are characterized within a still more general duality framework.
- There are non-trivial examples of various kinds.
- The bicrossproduct for groups with a compact open subgroup is somewhat special.
- It is expected to have an operator algebraic version, inspired by the bicrossproduct examples.

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